2201

Exam 2010. FurtherLinear algebra. Solutions.

QUESTION 1.

(a) [unseen]

Eucledian division: $68 = 5 * 12 + 8$, then $12 = 1 * 8 + 4$ and then $8 = 2 * 4 + 0$ hence $hcf(68, 12) = 4$

By doing algoritthm in reverse, one finds:

$4 = 1 * 12 + (-1) * 8) = (-1) * 68 + 6 * 12 = 6 * 12 + (-1) * 68$ hence $h = -1$ and $k = 6$.

(b) [unseen] The equation $68x + 12y = 4$ has solutions in integers because 4 divides 4.

The general solution is $(-1 + 3n, 6 - 17n)$ where n runs through the integers.

The equation $68x + 12y = 6$ does not have solutions in integers because 4 does not divide 6.

(c) Chinese remainder theorem. [seen] Let m and n two coprime integers. For any integers x and y, there exists a unique $[z]$ in $\mathbb{Z}/mn$ such that $z \equiv x \bmod m$ and $z \equiv y \bmod n$.

For $m = 21$ and $n = 5$, we have $1 = 21 - 4 * 5$. [unseen] We take

$$z = 21 * 7 - 4 * 5 * 3 = 87$$

QUESTION 2.

(a) [seen] Eucledian division :For any f, g in $k[x]$ with $deg(f) \geqslant deg(g)$, there exist a unique pair (q, r) such that $f = gq + r$ with $deg(r) < deg(g)$.

Bézout's identity : there exist $h, k \in k[x]$ such that $hcf(f, g) = fh + gk$.

(b) [seen] f irreducible if for any g dividing f, g is either constant or equal to f.

Unique factorisation theorem : For any monic polynomial f in $k[x]$, there exist monic irreducible polynomials $p_1, \ldots, p_r$ such that

$$f = p_1 \cdots p_r$$

If $f = q_1 \cdots q_s$ with q_i monic irreducibles, then $s = r$ and (after reodering) $q_i = p_i$ for all is.

(c) [unseen] In $\mathbb{C}[x]$, $x^3 - 1 = (x-1)(x-\omega)(x-\omega^2)$ where $\omega = e^{\frac{2\pi i}{3}}$. Factors irreducible because they have degree one.

Now $(x-\omega)(x-\omega^2) = x^2 + x + 1$. This is irreducible in $\mathbb{R}[x]$ because has degree two and no real root. $(x-1)$ is irreducible in $\mathbb{R}[x]$ because has degree one. Hence, in $\mathbb{R}[x]$ the factorisation is $x^3 - 1 = (x-1)(x^2 + x + 1)$.

In $\mathbb{F}_3[x]$, $x^2 + x + 1 = (x-1)^2$ hence the factorisation is $(x-1)^3$. Factors irreducible because of degree one.

In $\mathbb{F}_2[x]$, $x^2 + x + 1$ is irreducible because degree two and has no root. Hence factorisation is $(x-1)(x^2 + x + 1)$.

(c) [seen] Minimal polynomial m_T : monic polynomial such that $m_T(T) = 0$ and for any non-zero f such that $f(T) = 0$, $\deg(f) \geqslant \deg(m_T)$.

Let f be such that $f(T) = 0$, then $\deg(f) \geqslant \deg(m_T)$. Eucledian division : $f = qm_T + r$ with $\deg(r) < \deg(m_T)$. By definition of the minimal polynomial $r = 0$ which implies that m_T divides f.

QUESTION 3.

(a) $V_{b_i}(\lambda_i) = \ker((T - \lambda_i Id)^{b_i})$.

(b) Let $v \in V_{b_i}(\lambda_i)$, then $(T - \lambda_i Id)^{b_i} v = 0$. Apply $T - \lambda_i Id$ and get $(T - \lambda_i Id)^{b_i+1} v = 0$. Hence

$$V_{b_i}(\lambda_i) \subseteq V_{b_i+1}(\lambda_i)$$

Let $v \in V_{b_i}(\lambda_i)$, then $(T - \lambda_i Id)^{b_i} v = 0$, hence

$$T(T - \lambda_i Id)^{b_i} v = (T - \lambda_i Id)^{b_i} T(v) = 0$$

, it follows that $T(v) \in V_{b_i}(\lambda_i)$. Hence

$$T(V_{b_i}(\lambda_i)) \subseteq V_{b_i}(\lambda_i)$$

(c) T is diagonalisable if V has a basis consisting of eigenvectors of T.
Criterion : T is diagonalisable if and only if $m_T(x) = (x - \lambda_1) \cdots (x - \lambda_r)$.

(d) For the first matrix one finds $m_A(x) = x(x-2)$. It is diagonalisable over $\mathbb{R}$ and $\mathbb{C}$ and $\mathbb{F}_3$ but not over $\mathbb{F}_2$ (over $\mathbb{F}_2$ $m_A(x) = x^2$). $2\times 4 = 8$

(e) Because the minimal polynomial is $x - \lambda$, T_1 is diagonalisable and in a certain basis is represented by the diagonal matrix with λ on the diagonal. In fact, this is true in *any* basis (change of basis is a conjugation by an invertible matrix and a diagonal matrix with same entries on the diagonal commutes with everything).
Same is true for T_2, so for any basis B, $[T_1]_B = [T_2]_B$

QUESTION 4.

(a) The charactristic polynomial is $(x-3)^2$, the minimal polynomial is the same.
$V_1(3)$ is spanned by $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $V_2(3) = k^2$.

Choose $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$A - 3I = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

We have $(A - 3I)v_2 = -v_1$.
$\{-v_1, v_2\}$ is a Jordan basis.
The Jordan normal form is

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

(b) (i)

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

(ii)

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix}$$

(iii)

$$\begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix}$$

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(iv)

$$\begin{pmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

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QUESTION 5.

(a) [seen] $f(v,w) = \frac{1}{2}(q(v+w) - q(v) - q(w))$ hence $q(v) = 0$ for all v implies $f = 0$. Therefore if f is non-zero, there is a v such that $q(v) \neq 0$. 2

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(b) [seen] Orthogonal basis : a basis $\{b_1, \ldots, b_n\}$ such that $f(b_i, b_j) = 0$ if $i \neq j$. 2
Existence of orthogonal basis: By induction on $\dim(V)$. If $\dim(V) = 1$, then take any basis (any non-zero vector).
Suppose true for all vector spaces of dimension $n-1$. Let V be of dimension n. If f is zero, then it's matrix is zero and nothing to prove. If f is non-zero, then there exists a vector such that $q(v) \neq 0$. (see question (a)) 7
Now $V = Span(v) \oplus \{v\}^{\perp}$, therefore $\dim\{v\}^{\perp} = n-1$. By induction assumption, one chooses an orthogonal basis $\{b_1, \ldots, b_{n-1}\}$ of $\{v\}^{\perp}$. Then $\{b_1, \ldots, b_{n-1}, v\}$is an orthogonal basis : $f(b_i, b_j) = 0$ if $i \neq j$ and $f(v, b_i) = 0$ for all i because $b_i \in \{v\}^{\perp}$.

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(c) [seen] Canonical form : matrix of f the form

$$\begin{pmatrix} I_r & 0 & 0 \\ 0 & -I_s & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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with respect to a certain (necessarily orthogonal) basis.
Signature is the pair (r,s) and the rank is $r+s$.
Let $\{b_1, \ldots, b_n\}$ be an orthogonal basis. Number b_is such that $b_1, \ldots, b_r$ are such that $f(b_i, b_i) > 0$; $b_{r+1}, \ldots, b_{r+s}$ such that $f(b_i, b_i) < 0$ for $i = r+1, \ldots, r+s$ and $f(b_i, b_i) = 0$ for $i > r+s$. Then replace the basis by 5

$$\{\frac{b_1}{\sqrt{f(b_1,b_1)}}, \ldots, \frac{b_r}{\sqrt{f(b_r,b_r)}}, \frac{b_{r+1}}{\sqrt{-f(b_{r+1},b_{r+1})}}, \ldots, \frac{b_{r+s}}{\sqrt{-f(b_{r+s},b_{r+s})}}, b_{r+s+1}, \ldots, b_s\}$$

In this basis, f is in the canonical form.

(d) [unseen] One finds, by doing row-column operations

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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The rignature is $(2,0)$, rank is two.

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QUESTION 6.

(a) The adjoint T^* is a linear map $T^*\colon V \longrightarrow V$ such that

$$< Tv, w >=< v, T^*w >$$

for all v, w in V.

If T' is another adjoint, then for all v, w

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$$< v, T^*w >=< v, T'w >$$

hence for all v, w

$$< v, (T^* - T')w >= 0$$

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Set $v = (T^* - T')w$, then

$$||(T^* - T')w|| = 0$$

for all w. It follows that $T^* = T'$.

(b) Let λ be an eigenvalue and v an eigenvector : $Tv = \lambda v$ and $v \neq 0$. Then

$$< Tv, v >= \lambda < v, v >=< v, Tv > \text{ because } T = T^* = \overline{\lambda} < v, v >$$

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Let v and w be eigenvactors corresponding to distinct eigenvalues λ and μ. Then

$$< Tv, w >= \lambda < v, w >=< v, Tw >= \overline{mu} < v, w >= \mu < v, w >$$

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(we have used that μ is real). It follows that $(\lambda - \mu) < v, w >= 0$ and $\lambda - \mu \neq 0$, hence $< v, w >= 0$.

(c) [unseen] For any v we have

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$$< Tv, Tv >=< v, T^*Tv >= 0$$

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hence $Tv = 0$ for any v, hence $T = 0$.